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1. The EPR "paradox" and Bell's inequalities.

Let us consider two spin 1/2 particles in the singlet state :

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+_z,\ -_z\rangle - |-_z,\ +_z\rangle)$$

The spin operators of the two particles are denoted as $\vec{\sigma}_1 = \vec{S}_1/(\hbar/2)$ and $\vec{\sigma}_2 = \vec{S}_2/(\hbar/2)$. The kets $|\epsilon_{1z},\ \epsilon_{2z}\rangle$ denote the joint eigenstates of $\sigma_{1z},\ \sigma_{2z}$, with the eigenvalues $\epsilon_{1z} = \pm 1,\ \epsilon_{2z} = \pm 1$. Let us assume that the two particles are spatially separated, so it is possible to measure independantly any spin component for each particle.

The eigenstates $|\pm_{\vec{u}}\rangle$ of a spin measurement along the unit vector $\vec{u}$, with polar angles θ and ϕ in spherical coordinates, are given by :

$$|+_{\vec{u}}\rangle = \cos(\theta/2)e^{-i\phi/2}|+_z\rangle + \sin(\theta/2)e^{i\phi/2}|-_z\rangle$$

$$|-_{\vec{u}}\rangle = -\sin(\theta/2)e^{-i\phi/2}|+_z\rangle + \cos(\theta/2)e^{i\phi/2}|-_z\rangle$$

1. Let us assume that the two particles move in opposite directions along the axis Oy. Calculate the expression of $|\psi\rangle$ in a basis $|\epsilon_{\vec{a}},\ \epsilon_{\vec{b}}\rangle$ of eigenstates $|\pm_{\vec{a}}\rangle$ for particle 1 et $|\pm_{\vec{b}}\rangle$ for particle 2, where $\vec{a}$ and $\vec{b}$ are two unit vectors in the plane xOz ($\phi_1 = \phi_2 = 0$), corresponding to the angles θ_1 et θ_2.

2. One performs an instantaneous measurement of $\sigma_{1a} = \vec{\sigma}_1.\vec{a}$ and $\sigma_{2b} = \vec{\sigma}_2.\vec{b}$ of the spin components along the unit vectors $\vec{a}$ and $\vec{b}$.

a. What are the 4 possibles results, and the probabilities of these results ?

b. What are the possibles results if one considers one particle only ? What are the probabilities of these results ?

c. What is the conditional probability to get the result $+1$ for particle 2, knowing that the measurement on particle 1 has given the result -1 ?

d. Assuming $\vec{a} = \vec{b}$, show that the measurement result for one spin is perfectly determined by the measurement result for the other spin. Find again this conclusion by using the "reduction of the wave packet" postulate. What can be said about the correlation between these measurements results ?

e. Show that the average value in state $|\psi\rangle$ of the product of results $E_Q(\vec{a}, \vec{b}) = \langle\psi|\sigma_{1a}\sigma_{2b}|\psi\rangle$ is given by : $E_Q(\vec{a}, \vec{b}) = -\vec{a}.\vec{b}$. What is the physical meaning of $|E_Q(\vec{a}, \vec{b})| = 1$?

3. Bell's inequalities (1964) : Einstein, Podolsky and Rosen argued in 1935 that when two particles are far enough of each other, the value of the spin for each particle must have a determined value, independant from any measurement realized on the other particle. Following this idea, John Bell built a very general model in which there might be a "hidden variable" λ, determining the

results ± 1 for the separate (and remote) measurements of σ_{1a} and σ_{1b}, using two "sign" functions :

$$A(\lambda, \vec{a}) = \pm 1, \quad B(\lambda, \vec{b}) = \pm 1$$

This model is "local", because $A(\lambda, \vec{a})$ does not depend on $\vec{b}$, neither $B(\lambda, \vec{b})$ on $\vec{a}$. Denoting $P(\lambda)$ the probability distribution of the variables λ, normalized as $\int d\lambda P(\lambda) = 1$, one has thus :

$$E_C(\vec{a}, \vec{b}) = \int d\lambda P(\lambda) A(\lambda, \vec{a}) B(\lambda, \vec{b})$$

(i) Considering the 4 vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$, $\vec{b}'$, one defines :

$$s(\lambda) = A(\lambda, \vec{a})B(\lambda, \vec{b}) - A(\lambda, \vec{a})B(\lambda, \vec{b}') + A(\lambda, \vec{a}')B(\lambda, \vec{b}) + A(\lambda, \vec{a}')B(\lambda, \vec{b}')$$

Show that $s(\lambda) = \pm 2$. Hint : write $s(\lambda)$ as

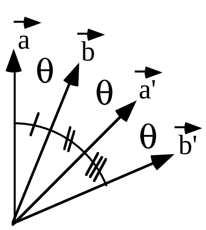
$$\left(A(\lambda, \vec{a}) + A(\lambda, \vec{a}')\right) B(\lambda, \vec{b}) - \left(A(\lambda, \vec{a}) - A(\lambda, \vec{a}')\right) B(\lambda, \vec{b}')$$

and look at the different possible values of $A(\lambda, \vec{a}) \pm A(\lambda, \vec{a}')$.

(ii) Using the above, demonstrate Bell's inequality : $|S_C| \leq 2$ with

$$S_C = E_C(\vec{a}, \vec{b}) - E_C(\vec{a}, \vec{b}') + E_C(\vec{a}', \vec{b}) + E_C(\vec{a}', \vec{b}')$$

4. Conflict between Quantum Mechanics and Hidden Variables Theories. Consider the choice of angles given on the figure below :



(i) Show that :

$$S_Q = E_Q(\vec{a}, \vec{b}) - E_Q(\vec{a}, \vec{b}') + E_Q(\vec{a}', \vec{b}) + E_Q(\vec{a}', \vec{b}')$$

can be written as : $S_Q = \cos(3\,\theta) - 3\cos(\theta)$.

(ii) Show that there is a conflict between the calculated predictions for S_Q and S_C for some values of θ. Conclusion ?

2. QND measurement of a spin component.

On wants to perform a measurement on a qubit "a" by using an indirect (rather than direct) measurement, called a "Quantum Non Demolition" (QND) measurement. For instance, if the qubit is a spin 1/2 particle, one will not use a Stern-Gerlach magnet, but rather get the spin "a" to interact with another spin "b" during a time τ, and read out the result on spin "b". For this purpose, let us denote $\vec{\sigma}_{a,b} = \vec{S}_{a,b}/(\hbar/2)$ the spin observables for the two qubits, and $|az : \pm 1\rangle$, $|bz : \pm 1\rangle$ the eigenstates of the observables σ_{az} and σ_{bz}. After the interaction, one measures (directly) the state of qubit b, and one wants to infer the states of qubit a.

1. Let us denote $|ax : \pm 1\rangle$ et $|ay : \pm 1\rangle$ the eigenstates of σ_{ax} et σ_{ay}. Write down these states in the basis $\{|az : \pm 1\rangle\}$. Write also the expression of $|ax : \pm 1\rangle$ as a function of $|ay : \pm 1\rangle$.

2. We assume that the qubits are motionless (e.g. they are trapped), and that their interaction is described by the hamiltonian $H_m = \hbar g\ \sigma_{az}\ \sigma_{bx}/2$, acting during a duration τ. All other effects will be negelected during the interaction time. Show that H_m, σ_{az} and σ_{bx} are commuting operators. Write down their eigenstates, and give the corresponding eigenvalues.

3. Let us assume that the initial state of the pair of qubits is $|\psi_+(0)\rangle = |az : +\rangle \otimes |by : +\rangle$, and adjust the duration of the interaction so that $g\tau = \pi/2$. Calculate the system's final state $|\psi(\tau)\rangle$. Answer the same question if the initial state is $|\psi_-(0)\rangle = |az : -\rangle \otimes |by : +\rangle$. Give an interpretation of these results by considering the expression of H_m and Bloch's sphere for the qubit b, in the two cases where the qubit a is in either of the two states $\{|az : \pm 1\rangle\}$.

4. Starting from the initial state $|\psi(0)\rangle = (\alpha|az : +\rangle + \beta|az : -\rangle) \otimes |by : +\rangle$, one measures the spin component of qubit b along Oz, after the interaction has been carried out and turned off.

What are the possible results, and what are their probabilities ? After this measurement, what can be said about the component along Oz for qubit a ? Justify the name "QND measurement" given to this kind of process.